Pre-Final Round 2019

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DATE OF RELEASE: 28. OCTOBER 2019

ABRAR

Important: Read all the information on this page carefully!

General Information

- Please read all problems carefully!
- We recommend printing this problem sheet. Use another paper to find the solutions to the problems and write your final solution (with steps) on the provided space below the problems.
- Please upload clear pictures of this problem sheet with your written answers. If you do not want to print this problem sheet, please clearly label the problems on your sheets.
- Typing the solution on a computer is possible. However, you do not receive extra points.
- The 10 problems are separated into three categories: 4x basic problems (A: three points), 4x advanced problems (B; four points), 2x special-creativity problems (C; six points).
- You receive points for the correct solution as well as for the performed steps. Example: Despite a wrong solution, if the described approach is correct you will still receive points.
- You can reach up to 40 points in total. You qualify for the final round if you reach at least 20 points (under 18 years) or 28 points (over 18 years).
- Please consider following notation that is used for the problems
  - $x, y \in \mathbb{R}$ denotes a real number, $n \in \mathbb{N}$ denotes a positive integer.
  - f, g, h denote functions. The domain and co-domain should follow from the context.
  - The "roots" of a function f are those x such that $f(x) = 0$.
  - $\pi = 3.141...$ denotes the circle constant and $e = 2.718...$ Euler's number.
- It is not allowed to work in groups on the problems. Help or assistance from teachers, friends, family, or the internet is prohibited. Cheating will result in immediate disqualification!

Solution Requirements

- You can upload your solution online via your status page: `https://iymc.info/en/login`
- Only upload one single PDF file! If you have multiple pictures, please compress them into one single file. Do not upload your pictures in a different format (e.g, no Word and Zip files).
- You can upload your PDF file with all solutions earlier than the day of the deadline. You can change your upload at any time as long as the deadline has not been reached.
- The deadline for uploading your solution is **Sunday 3. November 2019, 23:59 UTC+0**.
- The results of the pre-final round will be announced on Monday 11. November 2019.

Good luck!

Problem A.1

Find the area enclosed be these three functions: $f(x) = 1, \quad g(x) = x + 1, \quad h(x) = 9 - x$

Let, $y=1$ —— (i)
$y=x+1$ —— (ii)
$y=9-x$ —— (iii)

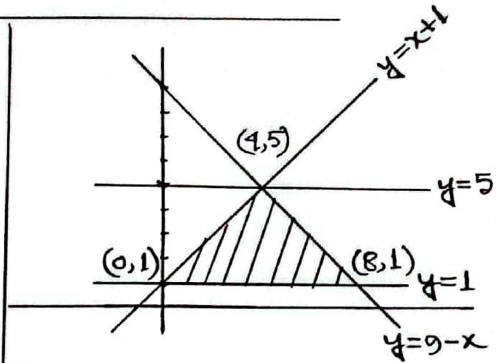
From (i) & (ii), $x+1=1 \Rightarrow x=0$
$\therefore (x,y)=(0,1)$

From (ii) & (iii), $x+1=9-x$
$\Rightarrow 2x=8$
$\Rightarrow x=4$
$\therefore (x,y)=(4,5)$

From (i) & (iii), $9-x=1$
$\Rightarrow x=8$
$\therefore (x,y)=(8,1)$

$$A = A_1 - A_2$$
$$= \int_1^5 (9-y)\,dy - \int_1^5 (y-1)\,dy$$
$$= \int_1^5 (9-y-y+1)\,dy$$
$$= \int_1^5 (10-2y)\,dy$$
$$= \left[10y - y^2\right]_1^5$$
$$= (50-25)-(10-1)$$
$$= 25-9$$
$$= \boxed{16 \text{ sq. unit}}$$

Ans : 16 sq. unit.



In triangle,
Area $= \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times |8-0| \times |5-1|$
$= \frac{1}{2} \times 8 \times 4$

$\boxed{\text{Area} = 16 \text{ sq. units}}$

Problem A.2

Find the roots of the function $f(x) = 3^x \cdot (\log_2(x) - 3)^5 \cdot e^{x^2-3x}$.

$f(x) = 3^x \cdot (\log_2 x - 3)^5 \cdot e^{x^2-3x} = 0$

When the product of factors equals to 0, at least one factor is 0.

$3^x = 0$
Here, $x \in \emptyset$

or, $(\log_2 x - 3)^5 = 0$
$\Rightarrow \log_2 x = 3$
$\Rightarrow x = 2^3 = 8$
$\Rightarrow \boxed{x=8}$

or, $e^{x^2-3x} = 0$
Here, $x \in \emptyset$

Ans : $x = 8$.

Problem A.3

Find the derivative $f'(x)$ of the function $f(x) = x^{\sin(x)}$.

Let, $f(x) = y = x^{\sin x}$

$\Rightarrow \ln y = \ln x^{\sin x} = \sin x \cdot \ln x$

$\Rightarrow \frac{1}{y} \cdot y' = \frac{\sin x}{x} + \ln x \cdot \cos x$

$\Rightarrow y' = y\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right]$

$$\boxed{\begin{aligned} \therefore f'(x) &= x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right] \\ &= x^{\sin(x)-1} \cdot \sin x + x^{\sin x} \cdot \ln x \cdot \cos x \end{aligned}}$$

(Ans.)

Problem A.4

Find the value of this expression for $n \to \infty$:

$$\left(\sqrt{1-\frac{1}{n}}\right)^n \cdot \sqrt{\left(1-\frac{1}{n}\right)^n}$$

Hint: You may use that $e^x = (1+\frac{x}{n})^n$ for $n \to \infty$.

Let, $y = \lim_{n\to\infty} \left(\sqrt{1-\frac{1}{n}}\right)^n \cdot \sqrt{\left(1-\frac{1}{n}\right)^n}$

$= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n$

$= \lim_{n\to\infty} \left[\left\{1+\left(-\frac{1}{n}\right)\right\}^{-n}\right]^{-1}$

Let, $z = -\frac{1}{n}$, then if $n \to \infty$, then $z \to 0^-$

$y = \lim_{z\to 0^-} \left[(1+z)^{1/z}\right]^{-1}$

$\Rightarrow y = \left[\lim_{z\to 0} (1+z)^{1/z}\right]^{-1}$ —— (1)

Let, $w = \lim_{z\to 0} (1+z)^{1/z}$

$\Rightarrow \ln w = \lim_{z\to 0} \frac{1}{z} \ln(1+z)$

$\Rightarrow \ln w = 1$

$\Rightarrow \log_e w = 1$

$\Rightarrow w = e$

$\therefore \lim_{z\to 0} (1+z)^{1/z} = e$

From (1),

$$\boxed{y = \left[\lim_{z\to 0} (1+z)^{1/z}\right]^{-1} = e^{-1} = \frac{1}{e}}$$

(Ans.)

Problem B.1

Find all positive integers n such that $n^4 - 1$ is divisible by 5.

Let, n be any positive integer. Since any integer is of the form $5q$ or, $5q+1$ or, $5q+2$ or, $5q+3$. We have $n^4-1=(n-1)(n+1)(n^2+1)$

If $n=5q$

$n^4-1=(5q-1)(5q+1)(25q^2+1)$

$= m$, which is not divisible by 5.

If $n=5q+1$

$n^4-1=(5q+1-1)(5q+2)(25q^2+10q+2)$

$=5q(5q+2)(25q^2+10q+2)$

$=5m$, divisible by 5

[where $m=q(5q+2)(25q^2+10q+2)$]

when ~~n=0~~ $n=1$; n^4-1 is divisible by 5

" $n=2$; n^4-1 " " " 5

" $n=3$; n^4-1 " " " 5

" $n=4$; n^4-1 " " " 5

" $n=5$; n^4-1 is not divisible by 5

If $n=5q+2$

$n^4-1=5(5q+1)(5q+3)(5q^2+4q+1)$

$=5m$, divisible by 5 [where $m=(5q+1)(5q+3)(5q^2+4q+1)$]

If $n=5q+3$

$n^4-1=5(5q+2)(5q+4)(5q^2+6q+2)$

$=5m$, divisible by 5 [where $m=(5q+2)(5q+4)(5q^2+6q+2)$]

$\therefore$ n^4-1 is only divisible by 5, when 'n' is not divisible by 5.

For $n \in \mathbb{Z}^+$

when $\frac{n}{5}=0$; then n^4-1 is not divisible by 5

" $\frac{n}{5}\neq 0$; " n^4-1 is divisible by 5.

Problem B.2

Prove the following inequality between the harmonic, geometric, and arithmetic mean with $x, y \geq 0$:

$$\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \leq \frac{x+y}{2}$$

Proof:

(i) Arithmatic mean $\geqslant$ Harmonic mean

$\frac{x+y}{2} \geqslant \frac{2}{\frac{1}{x}+\frac{1}{y}}$

$\Rightarrow \frac{x+y}{2} \geqslant \frac{2xy}{x+y}$

$\Rightarrow (x+y)^2 \geqslant 4xy$

$\Rightarrow (x+y)^2-4xy \geqslant 0$

$\Rightarrow (x-y)^2 \geqslant 0$

Since square is always $\oplus$ve, above statement is true.

(ii) G.M. $\leqslant$ A.M.

$\sqrt{xy} \leqslant \frac{x+y}{2}$

$\Rightarrow xy \leqslant \frac{(x+y)^2}{4}$

$\Rightarrow 4xy \leqslant (x+y)^2$

$\Rightarrow 0 \leqslant (x+y)^2-4xy$

$\Rightarrow 0 \leqslant (x-y)^2$

Since square is always positive, above statement is true.

(iii) G.M. $\geqslant$ H.M.

$\sqrt{xy} \geqslant \frac{2xy}{x+y}$

$\Rightarrow xy \geqslant \frac{4x^2y^2}{(x+y)^2}$

$\Rightarrow (x+y)^2 \geqslant 4xy$ [Dividing by xy on both sides]

$\Rightarrow (x+y)^2-4xy \geqslant 0$

$\Rightarrow (x-y)^2 \geqslant 0$

Since square is always $\oplus$ve, above statement is true.

Combining statements of (i), (ii), (iii) we get H.M. $\leqslant$ G.M. $\leqslant$ A.M. for $x, y \geqslant 0$

$$\therefore \frac{2}{\frac{1}{x}+\frac{1}{y}} \leqslant \sqrt{xy} \leqslant \frac{x+y}{2}$$

(Proved)

Problem B.3

Suppose you have to distribute the numbers $\{1, 2, 3, \ldots, 2n-1, 2n\}$ over n buckets. Show that there will always be at least one bucket with its sum of numbers to be $\geq 2n+1$.

Let's try to disprove it.

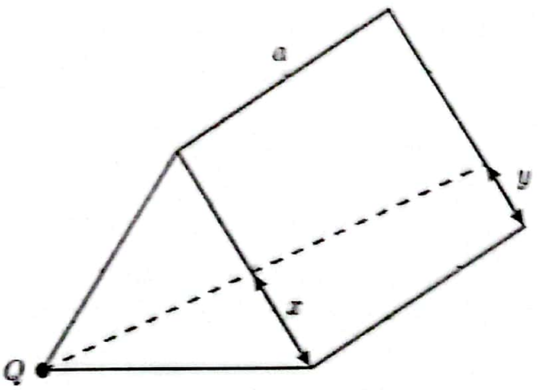
$\text{Total} = \frac{2n(2n+1)}{2} = 2n^2+n$

$\text{Average} = \frac{2n^2+n}{n} = 2n+1$

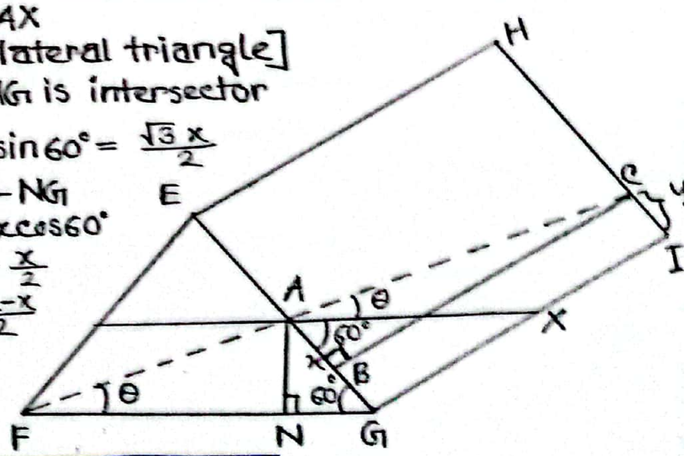
Therefore, sum of numbers equals to $2n+1$.
As we are disproving it, let's reduce the sum by 'x' from one bucket. So, we will have to place 'x' in another bucket. The sum in the 2nd bucket $= 2n+x+1$ which is greater than $2n+1$.
Or, we can think, to keep the sum less, there can be a maximum of $2n$ in each bucket.
So, total in 'n' bucket $= 2n \times n = 2n^2$
But given total is $2n^2+n$. So, it's not possible to disprove it. (Showed)

Problem B.4

Consider an equal-sided triangle connected to a square with side a (see drawing). A straight line from Q intersects the square at x and y. You have given x, find an equation for the intersection at $y(x)$.



We draw $BC \perp GE$. Let, $\angle CFG = \theta = \angle CAX$
We draw $AN \perp FG$. $\angle AGN = 60^\circ$ [$\because$ Equilateral triangle]
So, $\angle BAX = 60^\circ$ because $AX \parallel GN$ & AG is intersector



In right-angled $\triangle ABC$,

$\tan(\theta+60^\circ) = \frac{BC}{AB} = \frac{a}{x-y}$

$\Rightarrow \frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta} = \frac{a}{x-y}$

$\Rightarrow \frac{\sqrt{3}+\frac{\sqrt{3}x}{2a-x}}{1-\sqrt{3}\cdot\frac{\sqrt{3}x}{2a-x}} = \frac{a}{x-y}$

$\Rightarrow \frac{\sqrt{3}a}{a-2x} = \frac{a}{x-y}$

$\Rightarrow (2+\sqrt{3})x - a = \sqrt{3}y$

$\Rightarrow \boxed{y = \left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)x - \frac{a}{\sqrt{3}}}$

$AN = x\sin 60^\circ = \frac{\sqrt{3}x}{2}$

$FN = FG - NG$
$= a - x\cos 60^\circ$
$= a - \frac{x}{2}$
$= \frac{2a-x}{2}$

$\tan\theta = \frac{AN}{FN}$
$= \frac{\sqrt{3}x}{2a-x}$

$\boxed{\text{Ans: } y(x) = \left(1+\frac{2}{\sqrt{3}}\right)x - \frac{a}{\sqrt{3}}}$

Problem C.1

The sum of divisor function $\sigma(n)$ returns the sum of all divisors d of the number n:

$$\sigma(n) = \sum_{d|n} d$$

We denote N_k any number that fulfils the following condition:

$$\sigma(N_k) \geq k \cdot N_k$$

Find examples for N_3, N_4, N_5 and prove that they fulfil this condition.

If n is any number and $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$ then,

$$\sigma(n) = \frac{p_1^{e_1+1}-1}{p_1-1} \cdot \frac{p_2^{e_2+1}-1}{p_2-1} \cdots \frac{p_k^{e_k+1}-1}{p_k-1}$$

(i) If $k=1$, for all positive natural numbers, ~~N_k~~ N_1 is true.
If $N_1=1$, $\sigma(1)=1$
$\therefore \sigma(1) = 1.1 = 1.N_1$
So, N_1 can be $\{1,2,3,4,\ldots\}$

(ii) If $k=2$, then for $N_2 = 6,12,18,20,24,28,30,\ldots$ etc. it's true.
$\sigma(6) = 1+2+3+6 = 12$; $\sigma(6) \geq 2.6$
$\because \sigma(6) = 2N_2 = 12$
So, N_2 can be $\{6,12,18,20,24,28,30,36,\ldots\}$

(iii) If $k=3$, and $\boxed{N_3 = 120, 180, 240, 360, 420, 480, 504, 540, 600, 660, \ldots \text{etc.}}$ the condition is true.

For, $N_3=120$, $\sigma(120) = \sigma(2^3 \times 3^1 \times 5^1) = \frac{2^4-1}{2-1} \cdot \frac{3^2-1}{3-1} \cdot \frac{5^2-1}{5-1} = 360$

$\boxed{\sigma(120) = 3 \times 120 = 360 \text{, satisfies condition}}$

(iv) If $k=4$, and $\boxed{N_4 = 27720, 30240, 32760, 50400, 55440, 60480, 65520, 75600 \ldots}$ etc. then the condition is true.

For, $N_4 = 27720$,
$\sigma(27720) = \sigma(5 \times 3^2 \times 2^3 \times 7 \times 11) = \frac{2^4-1}{2-1} \times \frac{3^3-1}{3-1} \times \frac{5^2-1}{5-1} \times \frac{7^2-1}{7-1} \times \frac{11^2-1}{11-1} = 112320$

And, $27720 \times 4 = 110880$

So, $\boxed{\sigma(27720) > 110880 \text{, which satisfies the condition.}}$

(v) If $k=5$, and $\boxed{N_5 = 122522400, 147026880, 183783600, \ldots \text{etc}}$ then the condition is true.

For, $N_5 = 122522400$,
$\sigma(122522400) = \sigma(2^5 \times 3^2 \times 5^2 \times 7 \times 11 \times 13 \times 17) = \frac{2^6-1}{2-1} \cdot \frac{3^3-1}{3-1} \cdot \frac{5^3-1}{5-1} \cdot \frac{7^2-1}{7-1} \cdot \frac{11^2-1}{11-1} \cdot \frac{13^2-1}{13-1} \cdot \frac{17^2-1}{17-1}$

$= 614210688$

And, $122522400 \times 5 = 612612000$

$\boxed{\text{Since, } \sigma(122522400) > 612612000 \text{ So, the condition is satisfied.}}$

Problem C.2

This problem requires you to read following recently published scientific article:

Encoding and Visualization in the Collatz Conjecture.
George M. Georgiou, arXiv:1811.00384, (2019).
Link: `https://arxiv.org/pdf/1811.00384.pdf`

Please answer following questions related to the article:

(a) Explain the *Collatz conjecture* in your own words. Have we proven this conjecture?

Let, n be any positive integer. If n is ~~odd~~ even, we do $\frac{n}{2}$ & if n is odd, we do $3n+1$. The iteration of such function eventually reaches 1. It is a conjecture in mathematics that concerns a sequence defined, for any positive integer of n, the sequence will always reach 1. This conjecture is not proven.

(b) What is the $C(n)$ cycle and the $T(n)$ cycle of the number $n = 48$?

C(48) sequence is $48 \to 24 \to 12 \to 6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$
the sequence is caught in the cycle $1 \to 4 \to 2 \to 1$

T(48) sequence is $48 \to 24 \to 12 \to 6 \to 3 \to 5 \to 8 \to 4 \to 2 \to 1$
the sequence is caught in the cycle $1 \to 2 \to 1$

$C(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ 3n+1, & \text{if } n \text{ is odd} \end{cases}$

$T(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ \frac{3n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$

(c) Explain the meaning of $\sigma_\infty(n)$ and calculate $\sigma_\infty(104)$.

$\sigma_\infty(n)$ is the "Total Stopping Time" of n. It is the least applications of T that the sequence of iteration will reach 1 for first time. $\sigma_\infty(n) = \inf\{k : T^k(n) = 1\}$. The length of binary encoding is $\sigma_\infty(n)$.

For n = 104, binary encoding = 1110110111
$\therefore \boxed{\sigma_\infty(104) = 10}$

T(104):
$104 \to 52 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1$
Total 10 steps to reach 1.

(d) Find the binary encoding of $n = 32, 53, 80$ and explain why they all start with "111".

For n = 32, binary encoding = 11111
n = 53, binary encoding = 111011110
n = 80, binary encoding = 11101111

The binary encoding of all $n \geq 5$ starts with "111" because the iteration will end with $8 \to 4 \to 2 \to 1$ each time.

In binary encoding, if T(n) is even we put '1', if it's odd we put 0. From right to left the encoding is done.

(e) Make a drawing of the Collatz curve of $n = 2^{10} = 1024$.

T(1024) sequence $1024 \to 512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$
Binary encoding = 1111111111

Fig: Collatz curve of $n = 2^{10} = 1024$

(f) What is more common according to the data: r-curves with finite girth or acyclic r-curves?

According to data, in the range $n \leq 10^8$, there are only 40 r-curves with no cycles i.e. acyclic with the largest n = 308. up to $n \leq 10^8$, there are only 3 r-curves with finite girth $g(G(n)) > 4$ & those are for n = 273, 410 & 820. In all three cases $g(G(n)) = 12$.